This paper discusses transmit antenna subset selection schemes for downlink closed-loop multiple-input-multiple-output (MIMO) systems in multicell environments. A simplified-cooperative scheme is proposed to increase the sum of the mutual information of the links in all cells. The proposed simplified-cooperative scheme is realistic in that it only utilizes a limited amount of information from adjacent links, while the conventional cooperative scheme requires a large amount of information such as channel state information and MIMO transmit configurations of adjacent links. In spite of the use of a limited amount of information, the proposed scheme provides almost 90 percent of the average mutual information provided by the cooperative scheme. Furthermore, the proposed scheme shows significant enhancements over the competitive scheme which does not consider the effects of antenna subset selection on the performance of adjacent links.

I. INTRODUCTION

Information theoretic results have shown that the capacity of wireless communication systems can be greatly enhanced by deploying multiple antennas at both the transmitter and receiver [1]-[3]. Early research on multiple-input multiple-output (MIMO) systems was focused on a one-to-one link environment where only a single transmitter-receiver pair exists. In this scenario, the optimal transmit scheme maximizing capacity is known as waterfilling power allocation along with channel decoupling by singular value decomposition (SVD) [4]. The full channel state information (CSI) of the MIMO channel, however, should be fed back to the transmitter, which causes MIMO systems to have a large overhead and to be vulnerable to time variation of channel states. Various semi-optimal schemes requiring less feedback have been investigated, and transmit antenna subset selection is one of these [5], [6]. In this scheme, transmit power is equally distributed to the selected antennas. As a result, only the index of the selected subset is fed back to the transmitter, greatly reducing the feedback burden.

Recently there has been increasing interest in multicell systems which are composed of multiple transmitter-receiver links [7]-[9]. Since each link interferes mutually, the MIMO transmit configuration of one link affects the mutual information of the other links as well as that of its own. In this context, the optimal transmit scheme for open-loop MIMO multicell systems was discussed by Blum [8], and a transmit scheme for closed-loop MIMO multicell systems were investigated in another paper by Ye and Blum [9]. Based on [9], in the cooperative scheme in this paper, it is assumed that MIMO transmit configuration of each transmitter is determined to maximize the sum of the mutual information of all links in the multicell system. On the other hand, in the competitive scheme, each MS in every cell is assumed to compete with each other to increase the mutual information of its own link. In [9], the mutual information of the cooperative scheme is shown to be larger than that of the competitive scheme.

The cooperative scheme in [9] presents an upper limit of the mutual information for MIMO multicell systems. However, this is difficult to apply in a realistic environment. The first reason for this difficulty is that each transmitter is required to know the full CSI of the desired link for the channel decoupling by SVD. Second, each transmitter is required to know a large amount of information of adjacent links such as the CSI and MIMO transmit configurations. Each transmitter calculates the instantaneous mutual information of adjacent links as a function of its own MIMO transmit configuration.

In this paper, a simplified-cooperative scheme using a transmit antenna subset selection is proposed for the closed-loop downlink MIMO systems in multicell environments. In the proposed scheme, the mutual information of adjacent links is considered as well as that of the desired link. Contrary to the cooperative scheme, the amount of the required information is much reduced, since the average mutual information of adjacent links calculated from stochastic channel model is utilized instead of the instantaneous mutual information. As a result, the proposed scheme is easy to implement in realistic environments where available information is limited.

The rest of the paper is organized as follows: system models are presented in Section II; the simplified-cooperative scheme with limited feedback information is proposed in Section III; numerical results are presented in Section IV; and finally, conclusions are given in Section V.

The following notations are employed in this paper. The superscript $H$ and $T$ denote the transpose and transpose conjugate, respectively. $\det(A)$ represents the determinant of matrix $A$, and $I_M$ means the $M$ by $M$ identity matrix. $\max\{a,b\}$ and $\min\{a,b\}$ respectively denote the larger and smaller value between $a$ and $b$.

II. SYSTEM MODELS

A downlink MIMO multicell system comprised of $M$ cells is considered. It is assumed that a single MS is selected by user schedulers at the given time and frequency in each cell. The MS and base station (BS) are assumed to be equipped with $N_r$ receive antennas and $N_t$ transmit antennas, respectively. Then, the received signal vector $y_i$ at the $i$-th MS (the MS in the $i$-th cell) is modeled as

$$y_i = \sqrt{P_i}H_{i,i}x_i + \sum_{j \neq i} \sqrt{P_j}H_{i,j}x_j + n_i,$$  \hspace{1cm} (1)$$

where $P_i$ is the transmit power at the $i$-th BS (the BS in the $i$-th cell), and $H_{i,j}$ is the channel impulse response matrix of the link from the $j$-th BS to the $i$-th MS. $x_i$ is the transmit signal vector at
the \( i \)-th BS, and each element of \( \mathbf{x} \) is assumed to have Gaussian distribution. \( \mathbf{n} \) is the noise vector at the \( i \)-th MS of which entry is a complex Gaussian random variable with zero-mean and variance of \( \sigma_n^2 \). \( \mathbf{H}_{ij} \) is assumed to experience short-term fading \( \mathbf{G}_{ij} \) and long-term fading \( g_{ij} \) which is composed of path loss, and log-normal shadow fading. It may be expressed as
\[
\mathbf{H}_{ij} = g_{ij} \mathbf{G}_{ij},
\]
where \( g_{ij} \) is defined as
\[
g_{ij} = A \left( \frac{r_{ij}}{R} \right)^{\alpha} \frac{\gamma_i}{10^\frac{\gamma_i}{10}},
\]
where \( A \) is a constant and \( r_{ij} \) is the distance between the \( j \)-th BS and the \( i \)-th MS. \( R \) and \( \alpha \) is cell radius and path loss exponent, respectively. \( W_{ij} \) is a zero-mean Gaussian random variable with standard deviation of \( \sigma_n \), \( \mathbf{G}_{ij} \) is a \( N_x \) by \( N_t \) matrix, of which entries are independent zero-mean complex Gaussian random variables with unit variance. The power control is assumed to be employed in order to compensate for the long-term fading, and \( P_t \) is determined as
\[
P_t = \Gamma, \sigma_n^2 / g_{ij},
\]
where \( \text{SNR}_i \) means target signal to noise ratio (SNR).

For the MIMO system described above, the amount of the instantaneous mutual information of the transmitter-receiver link of the \( i \)-th cell is considered as the performance measure, which is given by [11]
\[
I_i = \log_2 \left( \det \left( \mathbf{P} \mathbf{H}_{ij} \mathbf{C} \mathbf{H}_{ij}^H + \mathbf{Q} \right) \right) - \log_2 \left( \det \left( \mathbf{Q} \right) \right),
\]
where \( \mathbf{C}_j \) is the correlation matrix of the transmit signal vector \( \mathbf{x} \) which is determined as the MIMO transmit configuration of the \( i \)-th BS, and \( \mathbf{Q} \) is the correlation matrix of the interference-plus-noise at the \( i \)-th MS, which is defined as
\[
\mathbf{Q} = \sum_{j \neq i} \mathbf{P} \mathbf{H}_{ij} \mathbf{C} \mathbf{H}_{ij}^H + \sigma_n^2 \mathbf{I}_{N_t}.
\]

In the following, \( \mathbf{H}_{ij}, \mathbf{Q} \), and \( P_t \) are assumed to be known at the \( i \)-th MS.

MIMO transmit configuration may be represented by the correlation matrix \( \mathbf{C}_j \), assuming equal power allocation between the selected antennas. It is defined as a diagonal matrix as follows:
\[
\mathbf{C}_j = \mathbb{E} \left\{ \mathbf{x} \mathbf{x}^H \right\} = \frac{1}{N_{ant}} \text{diag} \left( \delta_1, \delta_2, ..., \delta_{N_t} \right),
\]
where \( N_{ant} \) is the number of antennas within the selected subset of the \( i \)-th cell. \( \delta_n (n = 1, 2, \ldots, N_t) \) is one if the \( n \)-th transmit antenna is within the subset, or zero if not.

### III. Transmit Antenna Subset Selection Scheme

In [11], it was shown that the mutual information of the desired link decreases as the number of interfering antennas increases, when the amount of the intercell interference is assumed to be fixed. However, the effect of \( N_{ant} \), the number of the selected antennas of the desired BS, depends on the operating signal to interference plus noise ratio (SINR). According to the out analytical study which is omitted in this paper, in most realistic SINR regions the amount of mutual information increases as the number of desired antennas increases. Hence, if MIMO transmit configurations in each link are determined only for increasing its own mutual information, the performances of adjacent links may be sacrificed. The performance of the total systems may be reduced in multicell systems if only the link of each MS is considered, unlike the one-to-one link case.

In this section, transmit antenna subset selection schemes for multicell environments are investigated. At first, two different selection schemes proposed in [9] are reviewed briefly, then the simplified-cooperative scheme is proposed. The selection schemes in [9] were developed for generalized water-filling signaling. In this paper, however, they are applied to transmit antenna subset selection.

#### A. Competitive Scheme

As described in (5) and (7), antenna subset is selected according to \( C_n \), the correlation matrix of the input data streams, since mutual information is a function of correlation matrix. In the competitive scheme, an antenna subset that maximizes only the instantaneous mutual information of each link is selected. The antenna subset selection of the competitive scheme at the \( i \)-th link is expressed as follows:
\[
C_{\text{comp}} = \arg \max_{c_i} \{ I_i \}
\]
where \( C_{\text{comp}} \), correlation matrix which maximizes \( I_i \) is selected, assuming that \( I_i \) is a function of \( C_i \) as described in (5). Then, the transmit antenna subset that corresponds to this correlation matrix is selected. There is no cooperation among BSs in different cells; hence information of adjacent links is not required. However, competition between BSs may increase the effects of intercell interference, and optimizing the sum of the mutual information of the entire systems is not achievable.

#### B. Cooperative Scheme

In this scheme, antenna subset is selected to maximize the sum of the instantaneous mutual information of all the links. The antenna subset selection of the cooperative scheme is expressed as follows:
\[
C_{\text{coop}} = \arg \max_{c_i} \left\{ I_i + \sum_{j \neq i} I_j \right\}
\]
In contrast to the competitive scheme, not only \( I_i \) but also \( I_j \) the mutual information of adjacent link, is included to determine the transmit antenna subset for the \( i \)-th link. In other words, \( C_i \) that maximizes the sum of \( I_i \) and \( I_j \) is selected. From (5), (6), \( I_j \) may be expressed as
\[
I_j = \log_2 \left( \det \left( \mathbf{P} \mathbf{H}_{ij} \mathbf{C} \mathbf{H}_{ij}^H + \mathbf{Q} \right) \right) - \log_2 \left( \det \left( \mathbf{Q} \right) \right),
\]
\[
\mathbf{Q}_j = \mathbf{P} \mathbf{H}_{ij} \mathbf{C} \mathbf{H}_{ij}^H + \sum_{j \neq i} \mathbf{P} \mathbf{H}_{ij} \mathbf{C} \mathbf{H}_{ij}^H + \sigma_n^2 \mathbf{I}_{N_t}.
\]
The first term in (11) is associated with the selected antenna subset of the \( i \)-th link on the mutual information of the \( j \)-th link, and the second term is associated with that of the \( \ell \)-th link, respectively. As described in Fig. 1, the \( i \)-th link denotes the desired link of which antenna subset should be selected, and the \( \ell \)-th link denotes one of adjacent cells around the \( j \)-th link. With regards to the \( j \)-th cell, both the \( i \)-th and the \( \ell \)-th cell denote its adjacent cells, and are considered in calculating \( \mathbf{Q}_j \).

As shown in (10) and (11), to obtain \( I_j \), the additional information of adjacent links such as channel state information \( \mathbf{H}_{ij} \) and \( \mathbf{H}_{ij} \) transmit power \( P_i \) and \( P_j \), correlation matrices \( C_i \) and \( C_j \) are required. Unfortunately, it is nearly impossible to obtain this information in realistic environments.

#### C. Simplified-Cooperative Scheme

In this subsection, a simplified-cooperative scheme is proposed. The conventional cooperative scheme is unrealistic in that it re-
quires a large amount of instantaneous information of adjacent links. To overcome this, the average mutual information of adjacent links calculated from stochastic channel models is utilized instead of the instantaneous mutual information. As a result, the instantaneous information of adjacent links is not required for the proposed scheme.

The first step to develop the proposed scheme is a simplification of (10) and (11). In (10), the mutual information of the j-th adjacent link is formulated in terms of the selected subset for the i-th link. For simplification, it is assumed that adjacent BSs are equipped with a sufficiently large number of antennas and all the antennas are selected. Then, the first term in (10) and the second term in (11) are simplified to identity matrices, respectively, by the law of large number as follows;

\[
P_{P_j} C_j H_{i,j}^H = P_{g_j}, G_j, C_j G_j^H = \Gamma \sigma^2 I_{N_i},
\]

\[
P_{P_j} C_j H_{i,j}^H = P_{g_j}, G_j, C_j G_j^H = \Gamma \sigma^2 \frac{g_j}{g_{i,j}} I_{N_i}, i \neq j \text{ and } \ell \neq j.
\]

This simplification enables the selection of the preferable transmit antenna subset for the i-th link without the information of the instantaneous CSI \( G_{i,j} \) and \( G_{j,i} \), and correlation matrix \( C_j \) and \( C_i \) of the adjacent links. Applying (12) to (10) and (11), the instantaneous mutual information of the adjacent j-th link may be approximated as

\[
\begin{align*}
\hat{I}_j &= \log_2 \left \{ \det \left( \Gamma \sigma^2 I_{N_i} + P_{P_j} C_j H_{i,j}^H + \gamma I_{N_i} \right) \right \} \\
&\quad - \log_2 \left \{ \det \left( P_{P_j} C_j H_{i,j}^H + \gamma I_{N_i} \right) \right \},
\end{align*}
\]

where \( \gamma = \sigma^2 + \Gamma \sigma^2 \sum_{i,j \neq i,j, g_{i,j} / g_{i,j}} \).

And it is re-expressed as

\[
\hat{I}_j = N_i \log_2 \left( \gamma + \Gamma \sigma^2 \right) - N_i \log_2 \gamma
\]

\[
+ \log_2 \left \{ \det \left( I_{N_i} + \frac{P_{g_{i,j}}}{\gamma} G_{i,j} C_j G_j^H \right) \right \}
\]

\[
- \log_2 \left \{ \det \left( I_{N_i} + \frac{P_{g_{i,j}}}{\gamma} G_{i,j} C_j G_j^H \right) \right \}.
\]

Since \( C_j \) is a diagonal matrix of which element is \( 1/N_{a,k} \) or zero,

\[
G_{i,j} C_j G_j^H = 1/N_{a,k} G_{i,j} G_{j,i}^H,
\]

where \( G_{i,j} \) is \( N_i \) by \( N_{a,k} \) matrix, and the columns of \( G_{i,j} \) correspond to the selected transmit antennas.

The second step in developing the proposed scheme is an averaging of the mutual information of adjacent links. The third and fourth terms in (15) may be averaged for the given \( \gamma, g_{i,j}, P_i \) and \( N_{a,k} \), using the results of [12]. It may be expressed as

\[
E \{ \hat{I}_j | P_{g_{i,j}}, P_{N_{a,k}} \} = N_i \log_2 \left( \frac{1}{\gamma} \right) + \frac{P_{g_{i,j}}}{\gamma} \left( \frac{1}{\gamma} \right) - \frac{P_{g_{i,j}}}{\gamma} \left( \frac{1}{\gamma} \right),
\]

where

\[
f(\psi, N_{a,k}) = \log_2 \left( \sum_{n=0}^{N_{a,k}} \sum_{m=0}^{N_{a,k}} \left( \frac{2\pi \gamma}{2\pi \gamma + \psi} \right)^{n+m} \frac{E_{s\psi} \left( N_{a,k} \right)}{\psi} \right).
\]

In this equation, \( n \) and \( m \) are max\{\( N_i, N_{a,k} \)\} and min\{\( N_i, N_{a,k} \)\}, respectively. \( E_s(x) \) is exponential integral function defined as

\[
E_s(x) = \int_0^x e^{-t} \frac{dt}{t}.
\]

Equation (18) is important since it represents a closed-form expression of the averaging process. This formula is in terms of a finite sum of well-known special function defined as exponential integral function described in (19), hence, the output of (18) may be obtained without explicit numerical integrations. As described in (14), \( \gamma \) is a function of \( g_{i,j} \) and \( g_{j,i} \), the long-term fading component of adjacent link. \( g_{i,j} \) and \( g_{j,i} \) are random variables determined by shadow fading component and distance between MS and BS. The distribution of the distance may be obtained with the assumption that the position of each MS is uniformly distributed in a cell. Then, the distribution of \( \gamma \) and \( g_{i,j} \) may be calculated numerically, and the average mutual information is given in the form of a function of \( P_i \) and \( N_{a,k} \) as follows;

\[
E \{ \hat{I}_j | P_{g_{i,j}}, P_{N_{a,k}} \} = W_j + \tilde{I}_j (P_i, N_{a,k}),
\]

where

\[
W_j = N_i \log_2 \left( \frac{1}{\gamma} \right),
\]

\[
\tilde{I}_j (P_i, N_{a,k}) = \int_{g_{i,j}} \left( \frac{P_{g_{i,j}}}{\gamma} \right) p_{g_{i,j}} \left( g_{i,j} \right) dg_{i,j}.
\]

\( p_{g_{i,j}} \) and \( p_{g_{i,j}}(g_{i,j}) \) are the probability density functions of \( \gamma \) and \( g_{i,j} \), respectively. Through the averaging process, the need for the information of the instantaneous value of \( \gamma \) and \( g_{i,j} \) is eliminated, and the probability distribution functions of them are only required. They may be easily obtained through Monte Carlo simulations, and the integrals in (21) and (22) may be also calculated numerically.

By substituting the formula in (20) for \( I_j \) in (9), the antenna subset selection of the proposed scheme is defined as,

\[
C_{i,\text{proposed}} = \arg \max \left\{ I_j + F \left( |P_i, N_{a,k}| \right) \right\},
\]

where

\[
F \left( |P_i, N_{a,k}| \right) = \sum_{j=1}^{N_i} \left| W_j + \tilde{I}_j (P_i, N_{a,k}) \right|.
\]

\( F(P_i, N_{a,k}) \) is referred as the priority function, since the antenna subset with larger \( F(P_i, N_{a,k}) \) has higher probability to be selected.

The priority function represents the effects of transmit power and the number of selected antennas of the desired BS to MSs in adjacent links. The similar concept utilized in the cooperative scheme may be represented as \( I_j \) in (9), while the calculation of the priority function does not require the CSI, MIMO transmit configuration, or transmit power of adjacent links. The simplification and the averaging in developing the priority function may result in the decrease of mutual information. However, this decrease is not significant compared to the gain obtained by reducing system overhead, which will be shown in the next section.

The properties of the priority function may be explained using Fig. 2. This figure is obtained by Monte-Carlo simulations where the numbers of transmit and receive antennas are both 4, and path
loss exponent and log standard deviation of shadow fading are 3.7 and 8dB, respectively. In this figure, the solid lines are the values of approximated priority functions which are derived with simplification in (12), while the dotted lines represent the exact values of priority functions which are obtained without simplification.

Firstly, for the given transmit power, it is observed that value of priority function increases as the number of selected antennas, \( N_{\text{sub},i} \), decreases. This result is consistent with that of [11] that a smaller number of interfering antennas gives higher mutual information. Secondly, as the transmit power of the desired BS increases, the difference of the value of priority function between different \( N_{\text{sub},i} \) becomes larger. From these observations and equation (23), it is inferred that the antenna subset with small \( N_{\text{sub},i} \) has a greater chance to be selected as transmit power becomes higher. Thirdly, the difference between the exact and approximated values is not negligible, especially with low transmit power. However, the performance degradation due to this difference is not significant. It will be discussed in the next section.

Compared to the cooperative scheme, the proposed transmit antenna subset selection scheme is easy to implement. This is because the calculation of the priority function is yielded by exploiting only the stochastic channel model parameters of adjacent links, such as the log standard deviation of shadow fading and path loss exponent. The instantaneous information of adjacent links such as CSI and correlation matrices is not necessary. Even though numerical integrations are required in (21) and (22), the proposed scheme is still tractable, since the calculation of the priority function may be done initially once, or periodically with long time interval for every base station. Moreover, the proposed system is robust to the inevitable mismatch of the stochastic parameters and cell load, which will be shown in numerical results.

### IV. NUMERICAL RESULTS

In this section, the performance of the proposed transmit antenna subset selection is evaluated by Monte-Carlo simulations. For comparison, simulation results for the performance of the competitive and the cooperative scheme are also provided. In the simulations, the regular hexagonal cellular model that consists of a center-located desired cell and 60 adjacent cells is assumed, and the antenna subset is selected at each cell based on the same selection scheme. The path loss exponent \( \alpha \) and log standard deviation of shadow fading \( \sigma_\alpha \) are assumed to be 3.7 and 8dB, respectively. To ignore the effects of thermal noise, the transmit power is adjusted to achieve the target received SNR of 20dB, since the effects of intercell interference is the main focus of this paper. The positions of MSs are assumed to be uniformly distributed in a cell, and antenna subset selection schemes are assumed to be run by MSs in all cells. Simulation results are produced after 10 iterations of the adaptation process, since, according to the simulation study, each selection scheme is found to converge to the stationary points within 10 iterations with 90% probability.

The simulation results for the average mutual information and 5% outage mutual information are provided in Fig. 3, where \( N_s \) equals to \( N_r \). Not surprisingly, the cooperative scheme provides the best performance. It is noted that the differences of the average mutual information between the proposed and the cooperative scheme are only 10% and 15%, when the numbers of antennas are 4 and 8, respectively. Considering the difference of the required information between the proposed and the cooperative scheme, such amount of degradation is not significant. Compared with the competitive scheme, the proposed scheme achieves 21% and 35% higher average mutual information, and 75% and 70% higher outage mutual information, when the numbers of antennas are 4 and 8, respectively.

From Fig. 3, it is also observed that the performance loss by the simplification in (12) is not significant. In case that the numbers of transmit and receive antenna are both 4, the mutual information loss by simplification is only 2% on average and 8% in outage. This result implies that the performance of the proposed scheme is not sensitive to the variation of the priority function. The robustness of the proposed scheme to the mismatch of system parameters will be discussed with the results of Fig. 5.

Fig. 4 shows the mutual information according to the distance between MS and BS in the desired cell, when the positions of MS in adjacent cells are uniformly distributed. This is investigated to show the effects of transmit power on the proposed scheme, which was also investigated in Fig. 2. In this figure, the distance is normalized with cell radius \( R \). Because of the power control, the average mutual information of the competitive scheme is not much affected by distance. On the other hand, the average mutual information of the proposed scheme decreases with the distance. As MS is located closer to BS, the transmit power is controlled to be lower. Hence the difference of the value of priority function according to \( N_{\text{sub},i} \) is small, and the antenna selection is performed towards maximizing the mutual information of the desired link only. On the contrary, the MS at the cell boundary usually requires high transmit power, hence the number of the selected antennas makes comparatively large difference of the value of priority function. Even though performance tends to degrade at the cell boundary, the proposed scheme provides improved average mutual information compared to the competitive scheme, owing to the reduction of interfering signals.

In Fig.5, the robustness of the proposed scheme to the mismatch of the log standard deviation is quantified. It shows the mutual information in terms of the log standard deviation assumed in the proposed scheme, when the real log standard deviation is 8.0dB. It is observed that the proposed scheme is robust to the variations of modeled log standard deviation. It is also observed that the priority function with correct parameter does not provide the highest mutual information because of the simplification in yielding the priority function. However, the difference from the highest mutual information is within 3% on average.

### V. CONCLUSIONS

This paper discusses transmit antenna subset selection in downlink MIMO multicell systems. A simplified-cooperative scheme is proposed, which employs the average mutual information of adjacent links to select transmit antenna subset. In the proposed scheme, transmit antenna subset is selected considering the performance of adjacent links by applying the priority function, while additional overhead such as the instantaneous CSI and MIMO transmit configuration are not required. The proposed scheme provides almost 90% of the average mutual information achieved by the cooperative scheme which needs a large amount of information of adjacent links. Compared to the competitive scheme, the proposed scheme can achieve almost 35% higher average mutual information, and almost 75% higher outage mutual information. The simulation results show that proposed scheme is resilient to the incorrectness of the system parameters.
REFERENCES


Fig. 1. Simplified model to evaluate mutual information of adjacent j-th cell (The i-th cell: desired cell, the j-th cell: cell to be evaluated, the k-th cell: the other adjacent cell).

Fig. 2. The value of priority function according to transmit power $(N_r = N_t = 4)$.

Fig. 3. Mutual information as a function of the number of antennas.

Fig. 4. Average mutual information as a function of the normalized distance $(N_r = N_t = 4)$.

Fig. 5. Effect of log standard deviation mismatching $(N_r = N_t = 4)$.